

# Condensate density of interacting bosons: A functional renormalization group approach

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(Received 15 June 2009; revised manuscript received 29 October 2009; published 30 November 2009)

We calculate the temperature-dependent condensate density  $\rho^0(T)$  of interacting bosons in three dimensions using the functional renormalization group (FRG). From the numerical solution of suitably truncated FRG flow equations for the irreducible vertices we obtain  $\rho^0(T)$  for arbitrary temperatures. We carefully extrapolate our numerical results to the critical point and determine the order parameter exponent  $\beta \approx 0.32$  in reasonable agreement with the expected value 0.345 associated with the  $XY$ -universality class. We also calculate the condensate density in two dimensions at zero temperature using a truncation of the FRG flow equations based on the derivative expansion including cubic and quartic terms in the expansion of the effective potential in powers of the density. As compared with the widely used quadratic approximation for the effective potential, the coupling constants associated with the cubic and quartic terms lead to small corrections of the condensate density.

DOI: [10.1103/PhysRevE.80.051129](https://doi.org/10.1103/PhysRevE.80.051129)

PACS number(s): 05.30.Jp, 03.75.Hh, 05.10.Cc

## I. INTRODUCTION

The condensed phase of interacting bosons is often studied using Bogoliubov's celebrated mean-field approximation [1]. However, if one tries to go beyond the Bogoliubov approximation and includes fluctuation corrections perturbatively, some terms in the perturbation series for the single-particle Green's function diverge. The upper critical dimension for these divergencies is  $D=3$  at zero temperature and  $D=4$  at finite temperature. In the critical dimension the divergencies are logarithmic [2], whereas in lower dimensions one encounters even stronger power-law singularities. Physically, these divergencies arise due to the coupling of transverse fluctuations to longitudinal ones in the condensed phase; the gapless nature of the transverse fluctuations associated with the Goldstone modes then gives rise to singularities in the perturbation series of longitudinal correlation functions [3–8], which have to be resummed to all orders in perturbation theory to obtain meaningful results. The controlled calculation of physical properties of interacting bosons requires, therefore, nonperturbative methods.

Recently several authors have studied interacting bosons by means of the renormalization group [9–14], which is an efficient method to resum the perturbation series and remove the singularities encountered in finite order perturbation theory. While most calculations so far have focused on properties of the superfluid ground state [9–11] or on the single-particle spectral function at zero temperature [12,13], Floerchinger and Wetterich [14] have used the nonperturbative functional renormalization group (FRG) to calculate thermodynamic observables of the interacting Bose gas in three dimensions. They used a truncation of the formally exact FRG flow equation for the generating functional  $\Gamma$  of the irreducible vertices [15] based on the derivative expansion [16] retaining field gradients and density fluctuations in the expansion of the effective potential up to second order. In Sec. II we shall present an alternative derivation of the resulting finite temperature flow equations based on the vertex expansion [17]. In contrast to Ref. [14], we shall carefully analyze

the temperature dependence of the condensate density  $\rho^0(T)$  in the critical regime and extract the order parameter exponent  $\beta$  from the numerical solution of our truncated FRG flow equations. Our result  $\beta \approx 0.32$  is quite close to the expected value  $\beta \approx 0.345$  of the three-dimensional  $XY$ -universality class [18]. We therefore conclude that our simple truncation of the vertex expansion yields quantitatively accurate results for the condensate density for all temperatures including the critical regime.

In order to estimate the effect of higher order many-body interactions, we shall in Sec. III go back to the derivative expansion approach [14,16] and calculate the condensate density at vanishing temperature within a truncation to second order in the derivatives and to *fourth order* in the expansion of the effective potential  $U(\rho)$  in powers of density fluctuations  $\rho - \rho^0$ ,

$$U(\rho) \approx U^{(0)} + \sum_{k=2}^4 \frac{U^{(k)}}{k!} (\rho - \rho^0)^k. \quad (1)$$

We find that the coupling constants  $U^{(3)}$  and  $U^{(4)}$  flow to rather large values, which indicates that the true effective potential  $U(\rho)$  is nonanalytic at  $\rho = \rho_0$ . Nevertheless, if one rescales the coupling constants  $U^{(3)}$  and  $U^{(4)}$  with the proper power of the running cutoff  $\Lambda$  they flow to zero and hence appear to be irrelevant in the RG sense. As a result, the inclusion of  $U^{(3)}$  and  $U^{(4)}$ , which describe three- and four-body interactions, has only a rather small effect on the numerical value of the condensate density.

## II. FRG APPROACH TO THE CONDENSED BOSE GAS

### A. Vertex expansion of the FRG flow equation

The starting point of our investigation is the following Euclidean action describing bosons with mass  $m$  subject to a repulsive contact interaction  $u_0$ :

$$S[\bar{\psi}, \psi] = \int d^D r \int_0^\beta d\tau \left[ \bar{\psi}(\mathbf{r}, \tau) \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi(\mathbf{r}, \tau) + \frac{u_0}{2} (\bar{\psi}(\mathbf{r}, \tau) \psi(\mathbf{r}, \tau))^2 \right], \quad (2)$$

where the chemical potential  $\mu$  and the inverse temperature  $\beta=1/T$  are fixed. The spatial integrals should be regularized by means of a short-distance cutoff  $\Lambda_0^{-1}$ , which is related to the finite extent of the interaction or, for hard core bosons, to the size of the particles. Model (2) depends on three dimensionless parameters,

$$\tilde{\mu} = \frac{2m\mu}{\Lambda_0^2}, \quad (3a)$$

$$\tilde{T} = 2\pi \frac{2mT}{\Lambda_0^2}, \quad (3b)$$

$$\tilde{u}_0 = 2mu_0\Lambda_0^{D-2}, \quad (3c)$$

where at this point we do not specify the dimensionality  $D$  of the system, and the factor of  $2\pi$  in the definition of the dimensionless temperature  $\tilde{T}$  is introduced for later convenience. We focus on the condensed phase, where the global  $U(1)$  symmetry of action (2) is spontaneously broken and the field  $\psi$  has a finite expectation value  $\phi^0 = \langle \psi(\mathbf{r}, \tau) \rangle$ , which by

translational invariance is independent of space  $\mathbf{r}$  and imaginary time  $\tau$ . Without loss of generality, we choose  $\phi^0$  to be real.

To derive formally exact FRG flow equations for the one-particle irreducible vertices of our model, we add a cutoff-dependent regulator function  $R_\Lambda(\mathbf{k})$  to the inverse free propagator in the Gaussian part of action (2). In momentum-frequency space the inverse free propagator is then

$$G_{0,\Lambda}^{-1}(K) = i\omega - \epsilon_k + \mu - R_\Lambda(\mathbf{k}), \quad (4)$$

where  $i\omega$  is a bosonic Matsubara frequency,  $\epsilon_k = \mathbf{k}^2/2m$  is the free dispersion in momentum space, and  $K = (\mathbf{k}, i\omega)$  is a collective label. The regulator function  $R_\Lambda(\mathbf{k})$  should satisfy

$$R_\Lambda(\mathbf{k}) \sim \begin{cases} 0 & \text{for } \Lambda \rightarrow 0, \\ \infty & \text{for } \Lambda \rightarrow \infty, \end{cases} \quad (5)$$

so that the infrared cutoff  $\Lambda$  suppresses long-wavelength fluctuations and we recover our original model for  $\Lambda \rightarrow 0$ . For convenience we use the Litim regulator [19]

$$R_\Lambda(\mathbf{k}) = (1 - \delta_{\mathbf{k},0}) Z_\Lambda^{-1} (\epsilon_\Lambda - \epsilon_k) \Theta(\Lambda^2 - \mathbf{k}^2), \quad (6)$$

where the dimensionless wave-function renormalization factor  $Z_\Lambda$  is defined in Eq. (25) below. The cutoff-dependent irreducible vertices  $\Gamma_\Lambda^{(n,m)}(K'_1, \dots, K'_n; K_m, \dots, K_1)$  in the condensed phase are defined via the functional Taylor expansion of the corresponding generating functional [20] in powers of the fluctuations  $\delta\phi_K = \phi_K - \delta_{K,0}\phi^0$ ,

$$\Gamma_\Lambda[\bar{\phi}, \phi] = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} \int_{K'_1} \dots \int_{K'_n} \int_{K_m} \dots \int_{K_1} \delta_{K'_1 + \dots + K'_n, K_m + \dots + K_1} \Gamma_\Lambda^{(n,m)}(K'_1, \dots, K'_n; K_m, \dots, K_1) \times \delta\bar{\phi}_{K'_1} \dots \delta\bar{\phi}_{K'_n} \delta\phi_{K_m} \dots \delta\phi_{K_1}. \quad (7)$$

The derivative of the functional  $\Gamma_\Lambda[\bar{\phi}, \phi]$  with respect to the infrared cutoff  $\Lambda$  can be expressed in closed form in terms of a deceptively simple FRG flow equation [15–17], which is equivalent to an infinite hierarchy of integrodifferential equations for the cutoff-dependent vertices  $\Gamma_\Lambda^{(n,m)}(K'_1, \dots, K'_n; K_m, \dots, K_1)$ . Following Refs. [21–23], we fix the flowing order parameter  $\phi_\Lambda^0 = \langle \psi(\mathbf{r}, \tau) \rangle$  by demanding that the vertices  $\Gamma_\Lambda^{(1,0)}$  and  $\Gamma_\Lambda^{(0,1)}$  with a single external leg should vanish identically for any value of the cutoff  $\Lambda$ . Therefore, it is convenient to consider the chemical potential  $\mu$  as part of the interaction term [20,21]; i.e., we include it into the effective action  $\Gamma_\Lambda[\bar{\phi}, \phi]$ .

We are interested in the temperature dependent order parameter  $\phi_*^0 = \lim_{\Lambda \rightarrow 0} \phi_\Lambda^0$ , which determines the condensate density via  $\rho_*^0 = (\phi_*^0)^2$ . The exact FRG flow equation for the flowing order parameter  $\phi_\Lambda^0$  depends on the flowing normal and anomalous self-energies [21],

$$\Gamma_\Lambda^{(1,1)}(K, K) = \Sigma_\Lambda^K(K) - \mu, \quad (8)$$

$$\Gamma_\Lambda^{(0,2)}(K, -K) = \Gamma_\Lambda^{(2,0)}(-K, K) = \Sigma_\Lambda^K(K), \quad (9)$$

and on the four types of vertices with three external legs,  $\Gamma_\Lambda^{(3,0)}$ ,  $\Gamma_\Lambda^{(2,1)}$ ,  $\Gamma_\Lambda^{(1,2)}$ , and  $\Gamma_\Lambda^{(0,3)}$ . To calculate the order parameter we therefore need the flowing self-energies and the flowing three-legged vertices, whose flow equations depend again on higher order vertices with four and more external legs. To obtain a closed system of FRG flow equations, we shall use here the truncation proposed in Ref. [12], which amounts to the following parametrization of the nonzero vertices with three and four external legs:

$$\Gamma_\Lambda^{(2,1)}(K'_1, K'_2; K_1) = \phi_\Lambda^0 [u_\Lambda(K'_1) + u_\Lambda(K'_2)], \quad (10)$$

$$\Gamma_\Lambda^{(1,2)}(K'_1; K_2, K_1) = \phi_\Lambda^0 [u_\Lambda(K_1) + u_\Lambda(K_2)], \quad (11)$$

$$\Gamma_\Lambda^{(2,2)}(K'_1, K'_2; K_2, K_1) = u_\Lambda(K'_1 - K_1) + u_\Lambda(K'_2 - K_1). \quad (12)$$

The system of flow equations for the order parameter and the self-energies is closed by demanding that the momentum-

and frequency-dependent function  $u_\Lambda(K)$  is related to the flowing self-energies via

$$\Sigma_\Lambda^K(K) = \mu + \sigma_\Lambda(K) + \rho_\Lambda^0 u_\Lambda(K), \quad (13a)$$

$$\Sigma_\Lambda^A(K) = \rho_\Lambda^0 u_\Lambda(K), \quad (13b)$$

where  $\rho_\Lambda^0 = (\phi_\Lambda^0)^2$  is the flowing condensate density and  $\sigma_\Lambda(K)$  is another  $K$ -dependent function satisfying  $\sigma_\Lambda(0) = 0$ . Equations (10)–(12) relate the vertices with three and four external legs to the normal and anomalous components of the irreducible self-energy and thus close the FRG flow equations for the order parameter and the self-energies. Our parametrization of the self-energies given in Eqs. (13a) and (13b) is motivated by the pioneering insights gained by Nepomnyashchy and Nepomnyashchy [2], implying that only the contribution to the self-energy contained in the function  $u_\Lambda(K)$  exhibits a nonanalytic  $K$  dependence for  $\Lambda \rightarrow 0$ , while the contribution  $\sigma_\Lambda(K)$  remains analytic. As noted in Ref. [12], the above truncation satisfies the Hugenholtz-Pines relation [24]

$$\Sigma_\Lambda^K(0) - \Sigma_\Lambda^A(0) = \mu, \quad (14)$$

as well as the Nepomnyashchy identity [2]

$$\lim_{\Lambda \rightarrow 0} \Sigma_\Lambda^A(0) = 0, \quad (15)$$

which holds at finite temperature for  $D \leq 4$  and at zero temperature for  $D \leq 3$ , since in these cases

$$\lim_{\Lambda \rightarrow 0} u_\Lambda(0) = 0. \quad (16)$$

The truncation [Eqs. (10)–(12)] amounts to the following approximation for the generating functional  $\Gamma_\Lambda[\bar{\phi}, \phi]$  defined in Eq. (7):

$$\Gamma_\Lambda[\bar{\phi}, \phi] = \Gamma_\Lambda^{(0)} + \int_K \bar{\phi}_K \sigma_\Lambda(K) \phi_K + \frac{1}{2} \int_K \delta \rho_K u_\Lambda(K) \delta \rho_{-K}, \quad (17)$$

where  $\delta \rho_K = \int_Q \bar{\phi}_Q \phi_{Q+K} - \delta_{K,0} \rho_\Lambda^0$  are the Fourier components of the fluctuating density  $\rho(X) - \rho_\Lambda^0$  [25], and  $\Gamma_\Lambda^{(0)}$  is an interaction correction to the grand canonical potential in units of the temperature. As shown in Ref. [12], for  $\Lambda \rightarrow 0$  the function  $u_\Lambda(K)$  develops a nonanalytic dependence on  $K$ , so that the corresponding effective potential in real space and imaginary time is nonlocal.

To further simplify the FRG flow equations, we shall replace on the right-hand sides of the flow equations,

$$u_\Lambda(K) \rightarrow u_\Lambda(0) = u_\Lambda. \quad (18)$$

Within this truncation, the FRG flow equation for the condensate density  $\rho_\Lambda^0 = (\phi_\Lambda^0)^2$  reduces to

$$\partial_\Lambda \rho_\Lambda^0 = \int_Q [2\dot{G}_\Lambda^K(Q) + \dot{G}_\Lambda^A(Q)], \quad (19)$$

while the normal and anomalous components of the self-energy satisfy [12]

$$\begin{aligned} \partial_\Lambda \Sigma_\Lambda^K(K) = & 2u_\Lambda \int_Q \{ \dot{G}_\Lambda^K(Q) + \dot{G}_\Lambda^A(Q) \} - 4u_\Lambda^2 \rho_\Lambda^0 \int_Q \{ \dot{G}_\Lambda^K(Q) \\ & \times [G_\Lambda^K(Q+K) + G_\Lambda^K(Q-K) + G_\Lambda^K(-Q+K) \\ & + 2G_\Lambda^A(Q-K)] + 2\dot{G}_\Lambda^A(Q)[G_\Lambda^A(Q+K) \\ & + G_\Lambda^K(Q+K)] \}, \end{aligned} \quad (20)$$

$$\begin{aligned} \partial_\Lambda \Sigma_\Lambda^A(K) = & 2u_\Lambda \int_Q \dot{G}_\Lambda^K(Q) - 4u_\Lambda^2 \rho_\Lambda^0 \int_Q \{ \dot{G}_\Lambda^K(Q)[G_\Lambda^K(Q+K) \\ & + G_\Lambda^K(Q-K) + G_\Lambda^A(Q+K) + G_\Lambda^A(Q-K)] + \dot{G}_\Lambda^A(Q) \\ & \times [G_\Lambda^K(Q+K) + G_\Lambda^K(Q-K) + 3G_\Lambda^A(Q+K)] \}. \end{aligned} \quad (21)$$

Here the single-scale propagators  $\dot{G}_\Lambda^K(K)$  and  $\dot{G}_\Lambda^A(K)$  are defined via the matrix equation

$$\begin{pmatrix} \dot{G}_\Lambda^K(K) & \dot{G}_\Lambda^A(K) \\ \dot{G}_\Lambda^A(K)^* & \dot{G}_\Lambda^K(-K) \end{pmatrix} = -\mathbf{G}_\Lambda(K) [\partial_\Lambda \mathbf{G}_{0,\Lambda}^{-1}(K)] \mathbf{G}_\Lambda(K), \quad (22)$$

where  $\mathbf{G}_\Lambda^{-1}(K) = \mathbf{G}_{0,\Lambda}^{-1}(K) - \Sigma_\Lambda(K)$ , and

$$\mathbf{G}_{0,\Lambda}(K) = \begin{pmatrix} G_{0,\Lambda}(K) & 0 \\ 0 & G_{0,\Lambda}(-K) \end{pmatrix}, \quad (23)$$

$$\Sigma_\Lambda(K) = \begin{pmatrix} \Sigma_\Lambda^K(K) & \Sigma_\Lambda^A(K) \\ \Sigma_\Lambda^A(K)^* & \Sigma_\Lambda^K(-K) \end{pmatrix}. \quad (24)$$

## B. Low-energy truncation and results

Following Ref. [12] we expand the analytic part  $\sigma_\Lambda(K)$  of the self-energy in powers of momenta and frequencies up to quadratic order,

$$\sigma_\Lambda(K) \approx i\omega(1 - Y_\Lambda) + \epsilon_k(Z_\Lambda^{-1} - 1) - (i\omega)^2 V_\Lambda. \quad (25)$$

Note that at the initial scale  $Y_{\Lambda_0} = Z_{\Lambda_0} = 1$  and  $V_{\Lambda_0} = 0$ , so that  $\sigma_{\Lambda_0}(K)$  vanishes. While at zero temperature it is essential to retain the couplings  $Y_\Lambda$  and  $V_\Lambda$  associated with the frequency dependence of the self-energy, these couplings are irrelevant at the critical fixed point associated with Bose-Einstein condensation, which is a classical phase transition. Since in this section we are interested in the finite temperature behavior of the condensate density, for our purpose it is sufficient to retain only the  $Z_\Lambda$  term in Eq. (25),

$$\sigma_\Lambda(K) \approx \sigma_\Lambda(\mathbf{k}, i\omega = 0) \approx \epsilon_k(Z_\Lambda^{-1} - 1), \quad (26)$$

and we further approximate

$$u_\Lambda(K) \approx u_\Lambda = r_\Lambda / \rho_\Lambda^0. \quad (27)$$

The normal and anomalous propagators are then simply

$$G_\Lambda^K(K) = \frac{-i\omega - Z_\Lambda^{-1} \epsilon_k - r_\Lambda + R_\Lambda(\mathbf{k})}{\omega^2 - r_\Lambda^2 + [Z_\Lambda^{-1} \epsilon_k + r_\Lambda - R_\Lambda(\mathbf{k})]^2}, \quad (28)$$

$$G_{\Lambda}^A(K) = \frac{r_{\Lambda}}{\omega^2 - r_{\Lambda}^2 + [Z_{\Lambda}^{-1}\epsilon_k + r_{\Lambda} - R_{\Lambda}(k)]^2}, \quad (29)$$

while the corresponding single-scale propagators are

$$\dot{G}_{\Lambda}^N(K) = -[\partial_{\Lambda}R_{\Lambda}(k)] \frac{r_{\Lambda}^2 + [-i\omega - Z_{\Lambda}^{-1}\epsilon_k - r_{\Lambda} + R_{\Lambda}(k)]^2}{\{\omega^2 - r_{\Lambda}^2 + [Z_{\Lambda}^{-1}\epsilon_k + r_{\Lambda} - R_{\Lambda}(k)]^2\}^2}, \quad (30)$$

$$\dot{G}_{\Lambda}^A(K) = -[\partial_{\Lambda}R_{\Lambda}(k)] \frac{2r_{\Lambda}[R_{\Lambda}(k) - Z_{\Lambda}^{-1}\epsilon_k - r_{\Lambda}]}{\{\omega^2 - r_{\Lambda}^2 + [Z_{\Lambda}^{-1}\epsilon_k + r_{\Lambda} - R_{\Lambda}(k)]^2\}^2}. \quad (31)$$

Substituting these expressions into Eqs. (19)–(21) and using the Litim regulator (6), the momentum integrations and frequency summations appearing in the FRG flow equations for the three independent couplings  $\rho_{\Lambda}^0$ ,  $u_{\Lambda}$ , and  $Z_{\Lambda}$  at finite temperature can all be performed analytically. In  $D$  dimensions the final result can be written in the following form:

$$\Lambda \partial_{\Lambda} \tilde{\rho}_{\Lambda}^0 = \frac{2K_D}{\pi D} \left( \frac{\Lambda}{\Lambda_0} \right)^{D+2} \left( 1 - \frac{\eta_{\Lambda}}{D+2} \right) Z_{\Lambda}^{-1} [\tilde{\beta}^3 S_{0,2}(\tilde{\beta} \tilde{E}_{\Lambda}) P_{\Lambda}^{(2)} - \tilde{\beta} S_{2,2}(\tilde{\beta} \tilde{E}_{\Lambda})], \quad (32)$$

$$\Lambda \partial_{\Lambda} \tilde{u}_{\Lambda} = -\frac{8K_D}{\pi D} \left( \frac{\Lambda}{\Lambda_0} \right)^{D+2} \left( 1 - \frac{\eta_{\Lambda}}{D+2} \right) \tilde{u}_{\Lambda}^2 Z_{\Lambda}^{-1} [\tilde{\beta}^5 S_{0,3}(\tilde{\beta} \tilde{E}_{\Lambda}) P_{\Lambda}^{(3)} - \tilde{\beta}^3 S_{2,3}(\tilde{\beta} \tilde{E}_{\Lambda}) P_{\Lambda}^{(1)}], \quad (33)$$

$$\Lambda \partial_{\Lambda} Z_{\Lambda} = \eta_{\Lambda} Z_{\Lambda}, \quad (34)$$

where the flowing anomalous dimension is

$$\eta_{\Lambda} = \frac{4K_D}{\pi D} \left( \frac{\Lambda}{\Lambda_0} \right)^{D+2} \tilde{\rho}_{\Lambda}^0 \tilde{u}_{\Lambda}^2 Z_{\Lambda}^{-1} \tilde{\beta}^3 S_{0,2}(\tilde{\beta} \tilde{E}_{\Lambda}). \quad (35)$$

Here  $K_D = 2^{1-D} \pi^{-D/2} / \Gamma[D/2]$  is the surface of the  $D$ -dimensional unit sphere divided by  $(2\pi)^D$ , the dimensionless inverse temperature  $\tilde{\beta} = 1/\tilde{T}$  is defined via Eq. (3b), and we have introduced dimensionless quantities

$$\tilde{\rho}_{\Lambda}^0 = \rho_{\Lambda}^0 \Lambda_0^{-D}, \quad (36a)$$

$$\tilde{u}_{\Lambda} = 2mu_{\Lambda} \Lambda_0^{D-2}, \quad (36b)$$

$$\tilde{r}_{\Lambda} = \tilde{\rho}_{\Lambda}^0 \tilde{u}_{\Lambda} = 2m\rho_{\Lambda}^0 u_{\Lambda} / \Lambda_0^2, \quad (36c)$$

$$\tilde{E}_{\Lambda} = \sqrt{\tilde{\epsilon}_{\Lambda}(\tilde{\epsilon}_{\Lambda} + \tilde{r}_{\Lambda})}, \quad (36d)$$

$$\tilde{\epsilon}_{\Lambda} = Z_{\Lambda}^{-1} \Lambda^2 / \Lambda_0^2, \quad (36e)$$

as well as dimensionless coefficients,

$$P_{\Lambda}^{(1)} = \frac{7}{4} \tilde{r}_{\Lambda} + \frac{11}{4} \tilde{\epsilon}_{\Lambda}, \quad (37a)$$

$$P_{\Lambda}^{(2)} = \tilde{r}_{\Lambda}^2 + \tilde{r}_{\Lambda} \tilde{\epsilon}_{\Lambda} + \tilde{\epsilon}_{\Lambda}^2, \quad (37b)$$

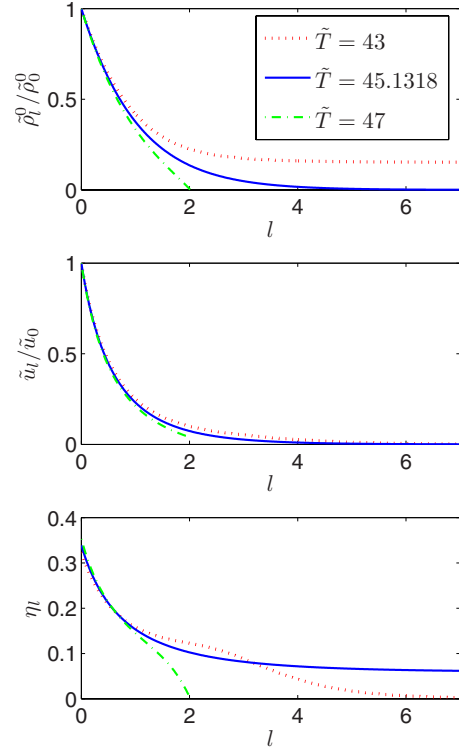


FIG. 1. (Color online) Typical FRG flow of the condensate density, the anomalous dimension and the dimensionless interaction  $\tilde{u}_l$  for  $D=3$  and  $\tilde{\mu}=\tilde{u}_0=1$ . Solid line: critical temperature ( $\tilde{T}_c = 45.1318$  for our choice of parameters); dashed line:  $\tilde{T}=47 > \tilde{T}_c$ ; dotted line:  $\tilde{T}=43 < \tilde{T}_c$ .

$$P_{\Lambda}^{(3)} = \tilde{r}_{\Lambda}^3 + \frac{3}{2} \tilde{r}_{\Lambda}^2 \tilde{\epsilon}_{\Lambda} + \frac{3}{4} \tilde{r}_{\Lambda} \tilde{\epsilon}_{\Lambda}^2 + \frac{5}{4} \tilde{\epsilon}_{\Lambda}^3. \quad (37c)$$

Finally, the dimensionless functions  $S_{k,l}(x)$  are defined in terms of the bosonic Matsubara sums,

$$S_{k,l}(x) = \sum_{n=-\infty}^{\infty} \frac{n^k}{(n^2 + x^2)^l}, \quad (38)$$

which can be expressed in terms of the Bose function and its derivatives.

The system of first-order differential equations given by Eqs. (32)–(34) can easily be solved numerically. It is convenient to consider quantities as functions of the logarithmic flow parameter  $l = -\ln(\Lambda/\Lambda_0)$ , renaming  $\tilde{\rho}_{\Lambda_0 e^{-l}}^0 \rightarrow \tilde{\rho}_l^0$ , and analogously for the other couplings. In Fig. 1 we show the typical RG flow of the couplings  $\tilde{\rho}_l^0$ ,  $\tilde{u}_l$ , and the flowing anomalous dimension  $\eta_l$  for three different temperatures (above, below, and at the critical temperature).

As expected, only at the critical temperature the flowing anomalous dimension has a finite limit (which can be identified with the critical exponent  $\eta$ ), while for  $T < T_c$  the condensate density approaches a nonzero value  $\tilde{\rho}_*^0$ . The temperature dependence of the corresponding order parameter  $\tilde{\phi}_*^0 = \sqrt{\tilde{\rho}_*^0}$  is shown in Fig. 2 for three different values of the interaction strength. In order to extract the order parameter exponent  $\beta$  from our numerical results shown in Fig. 2, one

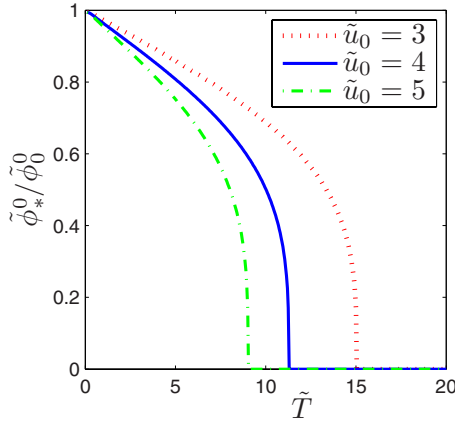


FIG. 2. (Color online) Temperature dependence of the order parameter  $\tilde{\phi}_*^0 = \sqrt{\tilde{\rho}_*^0}$  in three dimensions for  $\tilde{\mu}=1$  and three different values of the dimensionless bare interaction  $\tilde{u}_0$  defined in Eq. (3c).

should carefully fit the curves in a sufficiently small temperature interval below the critical temperature to a power law,

$$\tilde{\phi}_*^0 \propto (\tilde{T}_c - \tilde{T})^\beta. \quad (39)$$

Due to the lack of *a priori* knowledge about the proper interval for the power-law fit, and due to the finite accuracy of the numerical data, it is nontrivial to extract the critical exponent  $\beta$  from the numerical solutions of the FRG flow equations shown in Fig. 1. In fact, in a recent FRG calculation [14] of the temperature-dependent condensate density the critical exponent  $\beta$  was not determined apparently due to a lack of numerical accuracy. Here we present an extrapolation procedure which allows us to obtain the critical exponent  $\beta$  with high accuracy. The crucial point is that one should use a series of increasingly narrow intervals close to the critical point for the fitting procedure. Specifically, we use intervals of the form

$$I_z = [(1 - 2^{-z})\tilde{T}_c, \tilde{T}_c], \quad (40)$$

which are parametrized in terms of the zoom factor  $z$ . For increasing values of  $z$  we fit the data in the corresponding temperature interval  $I_z$  to power law (39) and extract  $\beta(z)$ . This procedure is illustrated in Fig. 3.

We then plot our results for  $\beta(z)$  as a function of  $z$  and extrapolate for  $z \rightarrow \infty$ . As shown in Fig. 4, the dependence of  $\beta(z)$  on the zoom factor  $z$  can be described by

$$\beta(z) \approx \beta[1 - e^{-\alpha(z+\gamma)}], \quad (41)$$

with some nonuniversal numbers  $\alpha$  and  $\gamma$ .

The critical exponent  $\beta$  can then be identified with  $\beta = \lim_{z \rightarrow \infty} \beta(z)$ . Given the simplicity of our truncation our final result  $\beta \approx 0.32$  is in reasonable agreement with the accepted value  $\beta \approx 0.345$  of the three-dimensional XY-universality class [18]. We have checked that our extrapolated result for  $\beta$  is independent of the various nonuniversal parameters of our model such as  $\Lambda_0$  or the value of  $\tilde{u}_0$ . We therefore believe that, in spite of its simplicity, our truncation of the vertex expansion yields accurate results for the

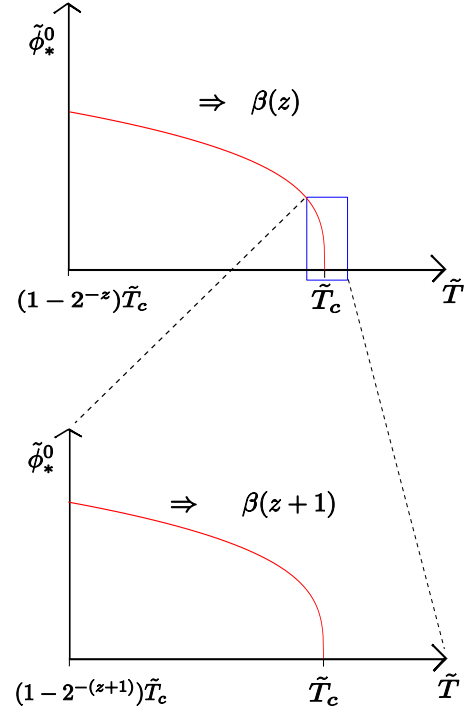


FIG. 3. (Color online) Iterative procedure to extract the order parameter exponent  $\beta$  from the FRG results. We fit our numerical FRG results in a series of intervals  $I_z$  given by Eq. (40) to power law (39) and thus determine the critical exponent  $\beta(z)$  for a given value of the zoom factor  $z$ . The  $z$  dependence of  $\beta(z)$  is then extrapolated for  $z \rightarrow \infty$  as shown in Fig. 4.

condensate density for all temperatures. We can estimate another critical exponent by identifying  $\eta = \lim_{l \rightarrow \infty} \eta_l$  with the anomalous dimension. If we fine tune the dimensionless temperature to its critical value  $\tilde{T} = \tilde{T}_c$  (as shown in Fig. 1)  $\eta_l$  converges toward  $\eta \approx 0.059$  independently of the chosen nonuniversal parameters. This value is of the correct order of magnitude. An improved estimate for  $\eta$  would be obtained only if higher order terms of the local density interaction were kept [26].

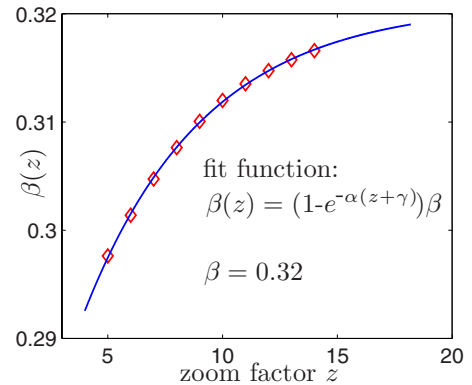


FIG. 4. (Color online) The dependence of  $\beta(z)$  on the zoom factor  $z$  can be described by the indicated fit function [see also Eq. (41)] with  $\alpha=0.1945$  and  $\gamma=8.545$ . The data are based on the numerical solution of the FRG flow equations for  $\tilde{u}_0 = \tilde{\mu}=1$ .



### III. DERIVATIVE EXPANSION WITH QUARTIC EFFECTIVE POTENTIAL

The truncation of the vertex expansion used in the previous section is equivalent to approximating the generating functional  $\Gamma_\Lambda[\bar{\phi}, \phi]$  of the irreducible vertices by Eq. (17). A weak point of this truncation is that it neglects many-body interactions involving more than two particles encoded in the irreducible vertices with more than four external legs. Within the framework of the vertex expansion it is rather difficult to take these vertices into account. In this section we shall therefore use the derivative expansion [16] with cubic and quartic terms in the effective potential to estimate the effect of higher order many-body interactions. For simplicity we consider in this section only the quantum renormalization of the condensate density at vanishing temperature.

If we approximate  $u_\Lambda(K) \approx u_\Lambda(0) \equiv u_\Lambda$  in Eq. (17) and use the low-energy expansion (25) for the analytic part  $\sigma(K)$  of the self-energy, then our truncated vertex expansion of Sec. II B amounts to the following approximation for the generating functional of the irreducible vertices:

$$\begin{aligned} \Gamma_\Lambda[\bar{\phi}, \phi] = & \int d^D r \int_0^\beta d\tau \left[ U_\Lambda(\rho(\mathbf{r}, \tau)) \right. \\ & + (1 - Y_\Lambda) \bar{\phi}(\mathbf{r}, \tau) \partial_\tau \phi(\mathbf{r}, \tau) \\ & \left. + (Z_\Lambda^{-1} - 1) \frac{|\nabla \phi(\mathbf{r}, \tau)|^2}{2m} + V_\Lambda |\partial_\tau \phi(\mathbf{r}, \tau)|^2 \right], \end{aligned} \quad (42)$$

where  $\rho(\mathbf{r}, \tau) = |\phi(\mathbf{r}, \tau)|^2$ , and we have written the local effective potential in the form

$$U_\Lambda(\rho) = U_\Lambda^{(0)} + \frac{U_\Lambda}{2} (\rho - \rho_\Lambda^0)^2, \quad (43)$$

with  $U_\Lambda^{(0)} = \Gamma_\Lambda^{(0)}/(\beta\mathcal{V})$  and where  $\mathcal{V}$  is the volume of the system. Note that at the initial RG scale we have  $Y_{\Lambda_0} = Z_{\Lambda_0} = 1$  and  $V_{\Lambda_0} = U_{\Lambda_0}^{(0)} = 0$ , so that  $\Gamma_{\Lambda_0}[\bar{\phi}, \phi]$  correctly reduces to the interaction part of the bare action minus the chemical potential term [20].

To investigate the effect of higher order many-body interactions, we now generalize the above ansatz by replacing the effective potential by a fourth-order polynomial in the density

$$U_\Lambda(\rho) = U_\Lambda^{(0)} + \frac{U_\Lambda^{(2)}}{2!} (\rho - \rho_\Lambda^0)^2 + \frac{U_\Lambda^{(3)}}{3!} (\rho - \rho_\Lambda^0)^3 + \frac{U_\Lambda^{(4)}}{4!} (\rho - \rho_\Lambda^0)^4. \quad (44)$$

In fact, for the truncated generating functional  $\Gamma_\Lambda[\bar{\phi}, \phi]$  of form (42) with an arbitrary local effective potential  $U_\Lambda(\rho)$  the exact FRG flow equation [15,16] implies the following partial differential equation for the effective potential [14]:

$$\begin{aligned} \partial_\Lambda U_\Lambda(\rho) = & \frac{K_D}{D} \left( 1 - \frac{\eta_\Lambda}{D+2} \right) \frac{\Lambda^{D+1}}{2mZ_\Lambda} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \\ & \times \frac{\rho U''(\rho) + U'(\rho) + V_\Lambda \omega^2}{[\rho U''(\rho) + U'(\rho) + V_\Lambda \omega^2]^2 - [\rho U'''(\rho)]^2 + Y_\Lambda^2 \omega^2}. \end{aligned} \quad (45)$$

At finite temperature, the frequency integral should be replaced by a bosonic Matsubara sum,  $\int \frac{d\omega}{2\pi} \rightarrow T \sum_\omega$ . To obtain an approximate solution of the partial differential Eq. (45), we expand  $U_\Lambda(\rho)$  in powers of  $\rho - \rho_\Lambda^0$ . The flowing condensate density  $\rho_\Lambda^0$  is then determined by

$$\left. \frac{\partial U_\Lambda(\rho)}{\partial \rho} \right|_{\rho_\Lambda^0} = 0, \quad (46)$$

and the expansion coefficients are

$$U_\Lambda^{(0)} = U_\Lambda(\rho_\Lambda^0), \quad U_\Lambda^{(k)} = \left. \frac{\partial^k U_\Lambda(\rho)}{\partial \rho^k} \right|_{\rho_\Lambda^0}. \quad (47)$$

Taking derivatives of Eq. (45), we obtain the flow equations for the condensate density and expansion coefficients,

$$\partial_\Lambda \rho_\Lambda^0 = - \frac{1}{U_\Lambda^{(2)}} \frac{\partial}{\partial \rho} [\partial_\Lambda U_\Lambda(\rho)] \Big|_{\rho_\Lambda^0}, \quad (48)$$

$$\partial_\Lambda U_\Lambda^{(k)} = \left\{ \frac{\partial^k}{\partial \rho^k} [\partial_\Lambda U_\Lambda(\rho)] - \frac{U_\Lambda^{(k+1)}}{U_\Lambda^{(2)}} \frac{\partial}{\partial \rho} [\partial_\Lambda U_\Lambda(\rho)] \right\} \Big|_{\rho_\Lambda^0}. \quad (49)$$

The flow of the couplings  $Z_\Lambda$ ,  $Y_\Lambda$ , and  $V_\Lambda$  related to the single-particle Green's function can be derived by inserting our ansatz (42) into the exact FRG flow equation for  $\Gamma_\Lambda[\bar{\phi}, \phi]$  and comparing terms with the same number of gradients on both sides [15,16]. If we retain only the quadratic coupling  $U_\Lambda^{(2)} \equiv u_\Lambda$  and set  $Y_\Lambda = 1$  and  $V_\Lambda = 0$  we recover flow equations (32)–(35) obtained in Sec. II within the vertex expansion. If we retain in addition the cubic and quartic couplings  $U_\Lambda^{(3)}$  and  $U_\Lambda^{(4)}$  the resulting system of equations is rather lengthy. In this work we do not give these equations explicitly because their derivation is straightforward and they can only be analyzed numerically anyway; technical details can be found in Ref. [27]. Note that the couplings  $U_\Lambda^{(k)}$  and  $V_\Lambda$  are not dimensionless; for our numerical analysis it is convenient to work with the corresponding dimensionless couplings

$$\tilde{U}_\Lambda^{(k)} = 2m U_\Lambda^{(k)} \Lambda_0^{D(k-1)-2}, \quad \tilde{V}_\Lambda = \frac{\Lambda_0^2 V_\Lambda}{2m}. \quad (50)$$

For simplicity, we have explicitly solved the coupled flow equations for the seven dimensionless couplings  $\tilde{U}_\Lambda^{(2)}$ ,  $\tilde{U}_\Lambda^{(3)}$ ,  $\tilde{U}_\Lambda^{(4)}$ ,  $\tilde{\rho}_\Lambda^0$ ,  $Z_\Lambda$ ,  $Y_\Lambda$ , and  $\tilde{V}_\Lambda$  only at zero temperature. The typical RG flow for the coupling constants  $\tilde{U}_\Lambda^{(2)}$ ,  $\tilde{U}_\Lambda^{(3)}$ , and  $\tilde{U}_\Lambda^{(4)}$  in two dimensions is shown in Fig. 5. Obviously, during the RG flow the cubic and quartic couplings  $\tilde{U}_\Lambda^{(3)}$  and  $\tilde{U}_\Lambda^{(4)}$  become orders of magnitude larger than the quadratic coupling  $\tilde{U}_\Lambda^{(2)}$ .

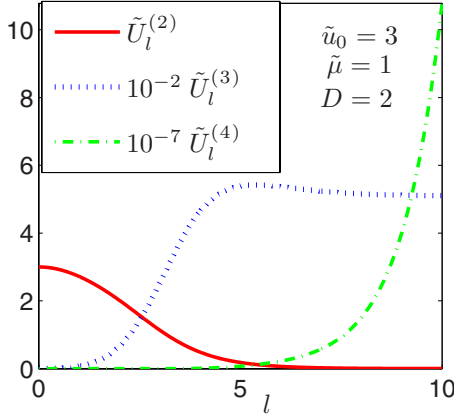


FIG. 5. (Color online) RG flow of the dimensionless couplings  $\tilde{U}_\Lambda^{(2)}$ ,  $\tilde{U}_\Lambda^{(3)}$ , and  $\tilde{U}_\Lambda^{(4)}$  defined in Eq. (50) for  $\tilde{u}_0=3$ ,  $\tilde{\mu}=1$ ,  $D=2$ , and  $T=0$ .

The behavior shown in Fig. 5 suggests that the true form of the effective potential cannot be approximated by a low order polynomial and might be traced to a nonanalyticity of  $U(\rho)$  at  $\rho=\rho_0$ . This is also the case in three dimensions where a similar calculation (not shown here) leads to a divergence of both coefficients  $\tilde{U}_\Lambda^{(3)}$  and  $\tilde{U}_\Lambda^{(4)}$  for  $\Lambda \rightarrow 0$ . Nevertheless, for the calculation of physical quantities like  $\rho_0$  it can still be reasonable to work with a polynomial approximation of  $U(\rho)$ , because the influence of the higher order terms is controlled by the IR-rescaled coupling constants  $\tilde{U}_\Lambda^{(k)}(\Lambda/\Lambda_0)^{D(k-1)-2}$ . As shown in Fig. 6 these couplings are nondivergent and in fact, for both  $D=2$  and  $D=3$ , flow to zero in the IR limit. The higher order terms of the density interaction can thus be regarded as irrelevant in the RG sense. This is reflected also by the fact that the large values of  $\tilde{U}_\Lambda^{(3)}$  and  $\tilde{U}_\Lambda^{(4)}$  (shown in Fig. 5) lead only to a small correction for the condensate density as shown in Fig. 7. Interestingly, the inclusion of cubic and quartic terms in the expansion of the effective potential renormalize the condensate density to larger values, so that the quadratic approximation overestimates the effect of fluctuations. It would be interesting to determine the true form of the effective potential in

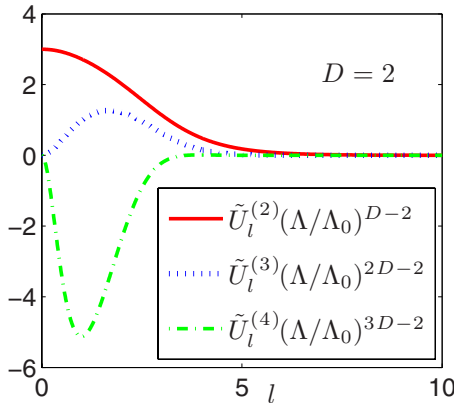


FIG. 6. (Color online) RG flows of the IR-rescaled, dimensionless couplings  $\tilde{U}_\Lambda^{(2)}(\Lambda/\Lambda_0)^{D-2}$ ,  $\tilde{U}_\Lambda^{(3)}(\Lambda/\Lambda_0)^{2D-2}$ , and  $\tilde{U}_\Lambda^{(4)}(\Lambda/\Lambda_0)^{3D-2}$  for  $\tilde{u}_0=3$ ,  $\tilde{\mu}=1$  in two dimensions.

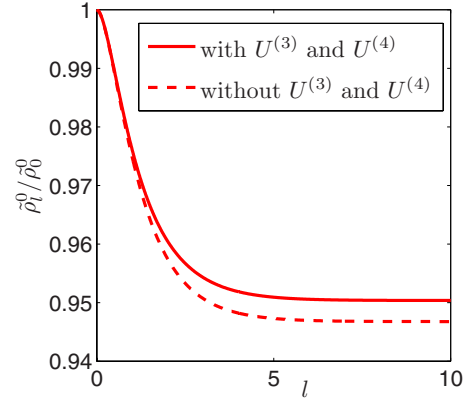


FIG. 7. (Color online) RG flow of the condensate density for  $\tilde{u}_0=3$ ,  $\tilde{\mu}=1$ ,  $D=2$ , and  $T=0$  with and without the cubic and quartic couplings  $\tilde{U}_\Lambda^{(3)}$  and  $\tilde{U}_\Lambda^{(4)}$ .

two and three dimensions by directly solving partial differential equation (45), which is, however, beyond the scope of this work.

#### IV. SUMMARY AND OUTLOOK

In summary, we have used two different truncation strategies for the formally exact FRG flow equation for the generating functional of the irreducible vertices to calculate the condensate density of the interacting Bose gas. Our first strategy presented in Sec. II is based on the truncated vertex expansion recently proposed in Ref. [12]. We have further simplified this truncation at finite temperature to obtain a closed system of flow equations for the condensate density, the effective interaction, and the wave-function renormalization factor at finite temperature. These flow equations are equivalent to the flow equations recently derived by Floerchinger and Wetterich within the derivative expansion [14]. From the numerical solution of these flow equations we have obtained a quantitatively accurate description of the critical regime. We have also developed an extrapolation procedure to extract the order parameter exponent  $\beta$  with high accuracy and moderate computational effort from the numerical solution of the FRG flow equations.

In order to estimate the renormalization of the condensate density by three-body and four-body interactions which are not included in our truncated vertex expansion of Sec. II, we have used in Sec. III the truncated derivative expansion in combination with a polynomial approximation for the effective potential and derived appropriate FRG flow equations at zero temperature. We have shown that the RG flow drives the cubic and quartic coefficients  $\tilde{U}_\Lambda^{(3)}$  and  $\tilde{U}_\Lambda^{(4)}$  in the expansion of the effective potential  $U_\Lambda(\rho)$  to rather large values. This indicates that a low order polynomial approximation is not appropriate to describe the true form of the effective potential in the condensed phase. Although the inclusion of the coefficients  $\tilde{U}_\Lambda^{(3)}$  and  $\tilde{U}_\Lambda^{(4)}$  leads only to a small positive correction of the condensate density, other physical observables might be more strongly affected by the three- and four-body interactions described by these couplings.

By using both the vertex expansion and the derivative expansion, we have clearly established the relation between these approximation strategies and have illustrated their advantages. In this work we have focused on the condensate density, because it appears naturally as one of the flowing couplings in the FRG flow equations. Our work can be extended in several directions: first of all, by keeping track of the field-independent part  $\Gamma_\Lambda^{(0)}$  of the generating functional in Eq. (17) (which can be identified with the interaction correction to the grand canonical potential per unit volume) one can also obtain the FRG flow of any thermodynamic observable of interest [14]. In the critical regime our numerical extrapolation procedure outlined in Sec. II should again be

useful to obtain quantitatively accurate results for other critical exponents. It would also be interesting to generalize the vertex expansion approach developed in Ref. [12] to finite temperatures and calculate the single-particle spectral function in the critical regime.

#### ACKNOWLEDGMENTS

We thank Andreas Sinner for discussions and gratefully acknowledge financial support by SFB/TRR49 and the DAAD/CAPES PROBRAL program. C.E. gratefully acknowledges support by the *Studienstiftung des Deutschen Volkes*.

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